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## Leaky Rigid Lid: New Dissipative Modes in the Troposphere

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### ABSTRACT

An effective boundary condition is derived for the top of the troposphere, based on a wave radiation condition at the tropopause. This boundary condition, which can be formulated as a pseudodifferential equation, leads to new vertical dissipative modes. These modes can be computed explicitly in the classical setup of a hydrostatic, nonrotating atmosphere with a piecewise constant Brunt–Väisälä frequency.

In the limit of an infinitely strongly stratified stratosphere, these modes lose their dissipative nature and become the regular baroclinic tropospheric modes under the rigid-lid approximation. For realistic values of the stratification, the decay time scales of the first few modes for mesoscale disturbances range from an hour to a week, suggesting that the time scale for some atmospheric phenomena may be set up by the rate of energy loss through upward-propagating waves.

### 1. Introduction

Much of our understanding of tropospheric dynamics is based on the concept of discrete internal modes. Internal gravity waves, such as those associated with convective systems, propagate at definite speeds, typically associated with the first to third baroclinic vertical modes, depending on the nature of the disturbance. Even though other effects such as nonlinearity, moist convection, and mean wind shear alter significantly the nature and speed of these waves, they remain nonetheless the dynamical backbone of the troposphere.

Yet discrete modes are the signature of systems of finite extent: a semi-infinite stratified atmosphere yields a continuum spectrum of modes, much as the Fourier transform in the infinite line, as opposed to the discrete Fourier series associated with finite intervals. This has led to arguments by R. Lindzen that these discrete tropospheric modes are just a fallacy of overly simplified theoretical models, and that the atmosphere “is characterized by a single isolated eigenmode and a continuous spectrum” (Lindzen 2003, p. 3009). On the other

hand, the troposphere does seem to operate on distinct discrete modes [see, e.g., Hayashi (1976) for an early reference], and many phenomena, some of which we mention below, have been modeled successfully on such basis.

Replacing the tropopause by a rigid lid where the vertical velocity must vanish is the simplest and most conventional way to obtain a discrete set of tropospheric modes with realistic values for their speed and vertical structure. Two justifications are typically provided for this approximation. One is that, for internal baroclinic waves, the oscillations at the free surface of stratified fluids have much smaller amplitude than those at internal isopycnals, as demonstrated in the famous experiment of Franklin with water and oil and manifested in the dead water phenomenon (Franklin 1905; Ekman 1904). This is indeed the basis for the rigid-lid approximation for the surface of the ocean, widely used for the study of its internal dynamics. Yet the tropopause is not the free surface between two fluids of very different density: it is not the density but its vertical derivative that has a strong gradient at the interface, typically modeled as a discontinuity.

The second justification for a rigid lid, more appropriate in the atmospheric context, is that the stratosphere, being much more strongly stratified than the

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troposphere, inhibits vertical motion. Yet the ratio of the stratification of the stratosphere to that of the troposphere, as measured by their representative Brunt–Väisälä frequencies  $N$ , is not infinite; in fact, it is rather close to 2. Can the rigid-lid approximation be justified under these circumstances? Do new effects come into play because of this finite ratio? These are the questions addressed in this article. As we shall see, the answer is affirmative to both, provided that the rigid-lid boundary condition is suitably modified. The three main new effects of taking into account the finite ratio of the Brunt–Väisälä frequencies are 1) there is a discrete set of modes, but they dissipate as they radiate a fraction of their energy into the stratosphere; 2) a slight change occurs in the speed and vertical structure of the modes; and 3) a new tropospheric mode appears with some barotropic characteristics and a mesoscale dissipation time scale of 1 h.

The need to impose boundary conditions at a finite height, such as the top of the troposphere or the upper end of a finite computational domain, has led to a variety of modeling approaches. The simplest boundary condition for a model of a finite atmosphere is a rigid lid, which has the vertical velocity set to zero at some finite height. Even though it is not completely justified on sound physical grounds, this boundary condition gives rise to one of the fundamental tools for understanding atmospheric dynamics—the rigid-lid modes. These have been used for a number of theoretical purposes, such as to study resonant interaction among waves (Raupp et al. 2008), to identify wave activity in the observational record (Haertel et al. 2008), and to study tropical–extratropical teleconnections (Kasahara and da Silva Dias 1986). The rigid lid is also an essential part of some modeling strategies for introducing moist dynamics into atmospheric models, projecting the dynamics onto the first few baroclinic rigid-lid modes, to yield a simplified vertical structure of the atmosphere with minimal vertical resolution (Majda and Shefter 2001; Khouider and Majda 2006). Prior attempts at improved boundary conditions relied on some form of radiation condition allowing all the internal gravity waves to leave the computational domain (Bennett 1976; Klemp and Durran 1983; Garner 1986; Purser and Kar 2001). On the other hand, some general circulation models, such as the Massachusetts Institute of Technology (MIT)’s, model the atmosphere as infinite—for instance, though the use of pressure coordinates.

In this paper we introduce two new results. First, we derive an effective boundary condition at the tropopause that permits modeling the troposphere in isolation from the rest of the atmosphere. Second, we compute “leaky” lid modes using the effective boundary

condition for a simple explicit background stratification. These modes have a novel feature: they decay with realistic time scales.

To derive the first result we perform a local calculation at the interface between the troposphere and the stratosphere, and obtain reflection and transmission coefficients. These characterize how much wave energy leaves the troposphere, and we use this information to construct the effective boundary condition. The most important assumption in our approach is that no waves return back from the stratosphere to the troposphere—though waves are allowed to reflect at the tropopause. As long as this assumption holds, we can substitute the stratosphere by the effective boundary condition. Ongoing research shows that our modeling approach can be extended to incorporate more complicated physics of the troposphere, such as Earth’s rotation and possibly convection and moisture. Yet in this article that introduces the leaky lid, we have purposely concentrated on the simplest scenario of dry irrotational linear waves in a nonrotating environment.

When we compute the modes with the effective boundary condition we obtain a qualitatively new result—modes with realistic decay time scales. The modes are computed in the special case in which the buoyancy frequency is a constant  $N = N_1$  in the troposphere, and has value  $N_2$  different from  $N_1$  at the bottom of the stratosphere. The temporal frequencies and speeds of the new modes are very close to those of the rigid lid; however, they have realistic decay time scales from 1 day to 1 week for the first two baroclinic modes in the mesoscales. High baroclinic modes have a slower decay, with a rate decreasing as approximately  $n^{-2}$ , where  $n$  is the vertical wavenumber. In addition to that, we find a new “zero” baroclinic mode, which is stationary and has the fastest decay time scale of 1 h, representing the fast adjustment of nonoscillatory disturbances. All but the  $n = 0$  modes have tilted vertical profiles. Even though the new decaying modes are computed for a finite troposphere bounded above by a lid, they preserve the dissipative features (through radiation) of models with an infinite atmosphere.

One of the remarkable features of the new model is that it only involves one parameter  $\alpha$ , which is a function of the ratio of the Brunt–Väisälä frequencies  $N_1/N_2$ , roughly  $\frac{1}{2}$  on Earth. By allowing  $N_1/N_2$  to change between 0 and 1, we obtain a one-parameter family of models of the atmosphere. In particular, we show that the rigid-lid approximation is correct when  $N_1/N_2 \rightarrow 0$ . In the limit  $N_1/N_2 \rightarrow 1$  the boundary between the domains disappears and the effective boundary condition reduces to a radiation condition.

The paper is organized as follows. Section 2 introduces the equations and the main assumptions. Section 3 computes the effective boundary condition. The new modes are introduced in section 4, and their features are discussed in section 5. Section 6 shows how to project initial conditions onto the new modes.

## 2. Basic equations

We consider the following simple model of a semi-infinite atmosphere through the linearized incompressible fluid equations in hydrostatic balance:

$$\begin{aligned}\rho_0 u_t + p_x &= 0, \\ p_z + g\rho &= 0, \\ \rho_t + w \frac{d\rho_0}{dz} &= 0, \\ u_x + w_z &= 0.\end{aligned}\quad (1)$$

Here,  $x$  and  $z$  are the zonal and vertical coordinates;  $u$  and  $w$  are the horizontal and vertical components of the velocity;  $p$  and  $\rho$  are the pressure and density perturbations from  $p_0(z)$  and  $\rho_0(z)$ , which are in hydrostatic balance; and  $g$  is the gravity constant. Note that  $\rho_0$  need not be independent of height in this approximation. For simplicity we consider here a 2D case. The extension to the nonrotating 3D case is shown in section 3.

Manipulating (1), one obtains an equation for the vertical velocity  $w$  alone:

$$\frac{(\rho_0 w_z)_{zz}}{\rho_0} + N^2 w_{xx} = 0, \quad N^2 = -g \frac{d\rho_0/dz}{\rho_0},$$

which, after the change of variables,

$$w = \phi / \sqrt{\rho_0}$$

becomes

$$\phi_{zzt} + \left( -\frac{N^4}{4g^2} + \frac{1}{2g} \frac{dN^2}{dz} \right) \phi_{tt} + N^2 \phi_{xx} = 0. \quad (2)$$

We nondimensionalize the equations using a typical depth of the troposphere  $\tilde{H}$ , a typical horizontal length scale  $\tilde{L}$ , and a reference value for the buoyancy frequency in the troposphere  $\tilde{N}$ . The typical scales for time and the horizontal and vertical velocities ( $\tilde{u}$  and  $\tilde{w}$ , respectively) are as follows:

$$\tilde{t} = \frac{\tilde{L}}{\tilde{N}\tilde{H}}, \quad \tilde{u} = \tilde{N}\tilde{H}, \quad \tilde{w} = \frac{\tilde{H}}{\tilde{L}}\tilde{u}. \quad (3)$$

Then (2) becomes

$$\phi_{zzt} + \left( -\varepsilon N^4 + \sqrt{\varepsilon} \frac{dN^2}{dz} \right) \phi_{tt} + N^2 \phi_{xx} = 0, \quad (4)$$

where  $\varepsilon = (\tilde{N}^2 \tilde{H} / 2g)^2$ . Typical values in the tropical troposphere are  $\tilde{H} = 16$  km and  $\tilde{N} = 0.01$  s<sup>-1</sup>, so  $\varepsilon \approx 0.006$  is small and we can replace (4) by

$$\phi_{zzt} + N^2 \phi_{xx} = 0, \quad (5)$$

except maybe at the tropopause, where  $dN^2/dz$  is large. We approximate this change in  $N$  over a small distance by a discontinuity, and impose the following jump conditions at the tropopause

$$[\phi] = 0, \quad [\partial_z \phi] = -\sqrt{\varepsilon} [N^2] \phi, \quad \text{at } z = 1. \quad (6)$$

Here, the square brackets denote the jump across the interface, which in the linear approximation is flat and at  $z = 1$ . These equations represent continuity of the vertical and horizontal velocities.

For realistic atmospheric values we have that  $\sqrt{\varepsilon} [N^2] \approx 0.23$ , which could be treated as a small parameter or not. We keep this term in the calculations in section 3 for the sake of generality, but neglect it in section 4 for simplicity when we compute the leaky modes in the special case of an atmosphere with piecewise constant buoyancy frequency.

## 3. The effective boundary condition

In this section we derive an effective boundary condition at the tropopause, as follows.

- (i) We look at a neighborhood of the interface, small enough that we can replace  $N$  by a constant on each side, and expand the solution in Fourier modes on each side of the interface.
- (ii) We impose a radiation condition: there should be no incoming waves from the stratosphere into the troposphere. This means that near the interface each mode consists of three components: an incoming wave from the troposphere, a reflected wave back into the troposphere, and a transmitted wave into the stratosphere. This scenario can be characterized by reflection and transmission coefficients that we calculate.
- (iii) We replace the stratosphere by an appropriate boundary condition at the tropopause that gives rise to the same reflection coefficient as computed in (ii).

By contrast, in the rigid-lid approximation, with  $\phi = 0$  at the tropopause, there is no transmitted wave and the reflection coefficient equals 1.

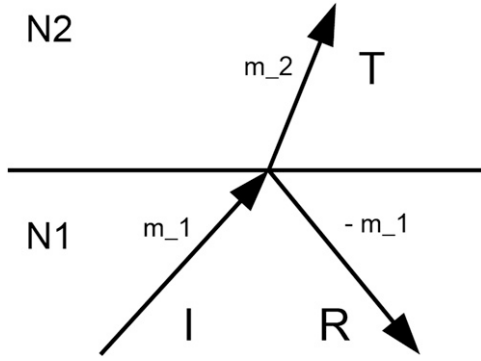


FIG. 1. Incoming, reflected, and transmitted waves at the interface between the troposphere ( $N = N_1$ ) and the stratosphere ( $N = N_2$ ), where  $N$  is the Brunt–Väisälä frequency. The wave amplitudes are  $I$ ,  $R$ , and  $T$  and the vertical wavenumbers are  $m_1$ ,  $-m_1$ , and  $m_2$ , respectively.

The first step in carrying out this program is to calculate the reflection and transmission coefficients. Since the problem is translation invariant, to simplify the notation in this section we consider the tropopause to be at  $z = 0$ ; in the rest of the paper we keep it at  $z = 1$ .

At the interface, where the Brunt–Väisälä frequency changes from  $N_1$  ( $z < 0$ ) to  $N_2$  ( $z > 0$ ), the incoming ( $I$ ), reflected ( $R$ ), and transmitted ( $T$ ) waves (Fig. 1)

$$\begin{aligned}\phi_I &= \exp\{i(k_I x + m_I z - \omega_I t)\}, \\ \phi_R &= R \exp\{i(k_R x + m_R z - \omega_R t)\}, \quad \text{and} \\ \phi_T &= T \exp\{i(k_T x + m_T z - \omega_T t)\}\end{aligned}$$

satisfy the dispersion relations

$$\omega_I = -\frac{N_1 |k_I|}{m_I}, \quad \omega_R = -\frac{N_1 |k_R|}{m_R}, \quad \omega_T = -\frac{N_2 |k_T|}{m_T}. \quad (7)$$

Here, the signs take into account the fact that the group velocity is positive for the incoming and transmitted waves and negative for the reflected wave.

At the interface, continuity of frequency and horizontal wavenumber imply

$$\omega_I = \omega_R = \omega_T = \omega, \quad k_I = k_R = k_T = k, \quad (8)$$

where  $k$  and  $\omega$  denote the common values of the corresponding parameters. The jump conditions (6) yield

$$T - (1 + R) = 0, \quad (9)$$

$$im_T T - (im_I + im_R R) = -\sqrt{\varepsilon}[N^2](1 + R), \quad (10)$$

which can be solved for the reflection coefficient  $R$  in terms of  $m_I$  and  $[N^2]$  using that  $m_R = -m_I$  and  $m_T = (N_2/N_1)m_I$  from (7) and (8).

Next, we construct an effective boundary condition that, when applied at the interface to upward-propagating waves, causes the same fraction of them to reflect downward, as if the stratosphere were there with  $N = N_2$ . We manipulate (10) into an expression involving only the parameters of the incoming and reflected waves

$$\begin{aligned}N_2\{-i \operatorname{sign}(k)\}ik(1 + R) - (im_I + im_R R)i\omega_I \\ = -\sqrt{\varepsilon}[N^2](1 + R)i\omega_I,\end{aligned}$$

which, translated from Fourier variables to physical space, becomes the effective boundary condition

$$N_2 H(\phi_x) + \phi_{tz} = \sqrt{\varepsilon}[N^2]\phi_I. \quad (11)$$

Here,  $H$  is the Hilbert transform, which in Fourier space is represented by multiplication by  $\{-i \operatorname{sign}(k)\}$ . The occurrence of the Hilbert transform here is not surprising, since it is an operator naturally associated with decay at infinity in elliptic problems and radiation conditions for hyperbolic ones. The Benjamin–Ono equation (Benjamin 1967; Ono 1975) is a classical example of the Hilbert transform occurring in the context of internal waves in stratified flows. Computing the Hilbert transform term does not add any numerical complications when using spectral methods for the horizontal coordinate, since  $\hat{H}(\phi_x) = |k|\hat{\phi}$ , where the hat represents the horizontal Fourier transform.

This effective boundary condition can be generalized to the nonrotating 3D case, which includes the meridional direction  $y$ . Then it has the form

$$N_2 \sqrt{-\Delta} \phi + \phi_{tz} = \sqrt{\varepsilon}[N^2]\phi_I. \quad (12)$$

Here,  $\sqrt{-\Delta}$  is the pseudodifferential operator replacing the  $H\partial_x$  of the 2D case; in Fourier space,  $\sqrt{-\Delta}$  is multiplication by  $\sqrt{k^2 + l^2}$ , where  $l$  is the meridional wavenumber.

#### 4. The leaky rigid-lid modes

Now we are ready to compute the leaky rigid-lid modes in the troposphere. We do so in the simplest scenario, neglecting the right-hand side of (11) and modeling the buoyancy frequency in the troposphere as a constant  $N_1$ , which jumps to  $N_2 > N_1$  at the tropopause. The equations to solve are

$$\begin{aligned}\phi_{tzz} &= -N_1^2 \phi_{xx}, \\ \phi_{tz} &= -N_2 H(\phi_x) \quad \text{at } z = 1, \\ \phi &= 0 \quad \text{at } z = 0.\end{aligned} \quad (13)$$



We look for solutions of the form  $\phi(t, x, z) = e^{ikx} e^{\lambda t} f(z)$ , and find that

$$f(z) = \sinh\left(\frac{kN_1}{\lambda} z\right).$$

The effective boundary condition at  $z = 1$  yields

$$\tanh\left(\frac{kN_1}{\lambda}\right) = -\frac{N_1}{N_2} \text{sign}(k).$$

Therefore,

$$\lambda = \lambda_n(k) = -\frac{|k|N_1}{\tanh^{-1}(N_1/N_2)} = \frac{|k|N_1}{-\alpha + i\pi n}, \quad (14)$$

where  $\alpha$  is the principal (real) value of  $\tanh^{-1}(N_1/N_2)$  and  $i\pi n$ , with  $n$  integer, arises from the periodicity of  $\tanh$  in the complex plane. For a realistic atmosphere  $N_1/N_2 \approx 1/2$ , therefore  $\alpha \approx 1/2$ . The decay rate and frequency of the modes are given respectively by

$$\text{Re}(\lambda_n) = -\frac{|k|N_1\alpha}{\alpha^2 + (\pi n)^2} \sim -\frac{|k|N_1\alpha}{\pi^2 n^2} \quad \text{as } n \rightarrow \infty, \quad (15)$$

$$\text{Im}(\lambda_n) = \frac{|k|N_1\pi n}{\alpha^2 + (\pi n)^2} \sim \frac{|k|N_1}{\pi n} \quad \text{as } n \rightarrow \infty. \quad (16)$$

The corresponding vertical structure for the leaky rigid-lid modes is given by

$$f_n(z) = -\sinh(\alpha z) \cos(\pi n z) + i \cosh(\alpha z) \sin(\pi n z). \quad (17)$$

In terms of this, each mode of the solution takes the form

$$\begin{pmatrix} u_n \\ w_n \\ \rho_n \\ p_n \end{pmatrix} = \begin{pmatrix} \frac{i}{k} \frac{df_n(z)}{dz} \\ f_n(z) \\ \frac{i}{\lambda_n} f_n(z) \frac{d\rho_0}{dz} \\ -\frac{i}{\lambda_n} \int^z f_n(s) ds \end{pmatrix} e^{ikx} e^{\lambda_n(k)t}. \quad (18)$$

## 5. Features of the leaky rigid-lid modes

In this section we show that, for realistic choices of  $N_1$  and  $N_2$ , and a horizontal length scale  $\tilde{L} = 1000$  km, the first three leaky-lid modes exhibit decay time scales of 1 h, 1.5 days, and about 1 week; for different horizontal scales, the corresponding decay times scale proportionally. The

fastest decay corresponds to  $n = 0$ —a mode that is absent under the conventional rigid-lid approximation. All the other modes converge to the rigid-lid modes in the limit of  $N_1/N_2 \rightarrow 0$ . Finally, all but the  $n = 0$  leaky modes have tilted vertical profiles.

### a. Decay time scales and slightly adjusted baroclinic speeds

From the decay rate in (15), the nondimensional decay time scales are given by

$$T_n = \frac{\alpha^2 + (\pi n)^2}{\alpha |k| N_1}. \quad (19)$$

For a realistic atmosphere with  $\alpha = 1/2$ ,  $\tilde{L} = 1000$  km,  $\tilde{H} = 16$  km, and  $\tilde{N} = 0.01 \text{ s}^{-1}$  (so the corresponding nondimensional  $N_1$  equals 1), this yields

$$T_0 = 1 \text{ h}, \quad T_1 = 1.5 \text{ days}, \quad T_2 = 5.7 \text{ days}, \quad (20)$$

where we have used the factor  $\tilde{L}/(\tilde{N}\tilde{H})$  to convert from the nondimensional units. The only variable in this factor that changes significantly from one atmospheric phenomenon to another is  $\tilde{L}$ ; the corresponding time scales change linearly with  $\tilde{L}$ . With an extratropical reference value  $\tilde{H} = 9$  km for the height of the tropopause, the time scales are 80% longer. The speeds of the modes are independent of  $\tilde{L}$  and are, for  $\tilde{H} = 16$  km,

$$v_0 = 0 \text{ m s}^{-1}, \quad v_1 = 49 \text{ m s}^{-1}, \quad v_2 = 25 \text{ m s}^{-1}. \quad (21)$$

The first and second baroclinic values are very close to the corresponding speeds of the rigid-lid modes, which for the same values of the dimensional parameters are 51 and 25.4  $\text{m s}^{-1}$ , respectively. From the temporal frequency in (16), for any  $\alpha$ , the leaky-lid frequencies and speeds approach those of the rigid-lid modes as  $n \rightarrow \infty$ . For the actual atmosphere the leaky-lid and rigid-lid frequencies and speeds are close for all  $n > 0$ , because  $\alpha \approx 1/2 \ll \pi$ .

### b. Leaky modes as a correction to the classic rigid-lid modes and reappearance of the rigid lid in the limit $N_1/N_2 \rightarrow 0$

One part of the vertical structure of the leaky-lid solution is the classic rigid-lid mode  $\sin(\pi n z)$  modified in amplitude by  $\cosh(\alpha z)$ , which lies between 1 and 1.2 for  $\alpha = 0.5$ . The new part of the mode is the term  $\sinh(\alpha z) \cos(\pi n z)$ , which is zero at the lower boundary  $z = 0$ , but always  $\sinh(\alpha)$  at the top boundary  $z = 1$ . These components and the full solution at time  $t = 0$  and  $kx = 1$  are presented in Fig. 2.

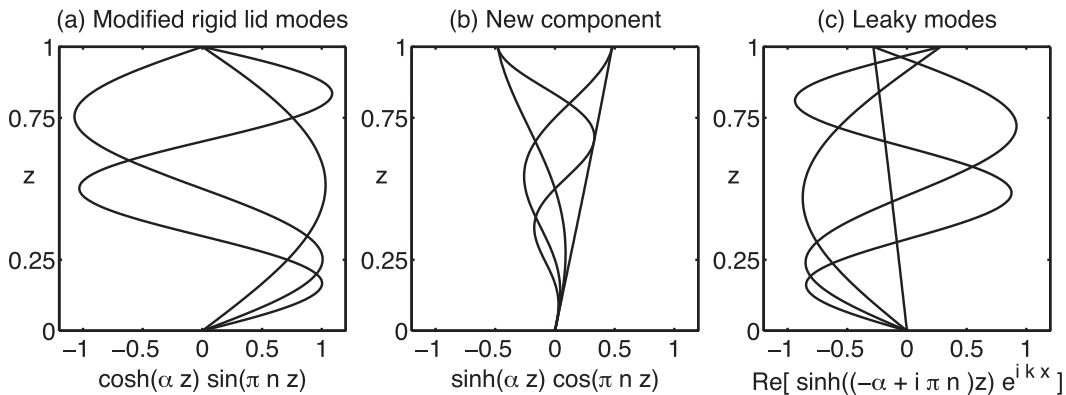


FIG. 2. Components of the leaky-lid modes ( $n = 0, 1, 2, 3$ ). (a) Component  $\cosh(\alpha z) \sin(\pi n z)$  is a slight modification of the classic rigid-lid modes. (b) Component  $\sinh(\alpha z) \cos(\pi n z)$  appears because of the radiative losses and gives the new  $n = 0$  mode not present in the rigid-lid case. (c) The full solution at  $t = 0$  and  $kx = 1$ .

A remarkable feature of the leaky-lid model is the appearance of the parameter  $\alpha = \tanh^{-1}(N_1/N_2)$ . The classic rigid-lid approximation is only valid in the limit  $\alpha \rightarrow 0$ , which corresponds to  $N_1/N_2 \rightarrow 0$ . Indeed, the vertical structure function in this limit becomes  $f(z) = \sin(\pi n z)$ , the decay time scales become infinite, and the leaky mode  $n = 0$ , described below, is no longer a non-trivial solution in either 2D or 3D. The frequency of the modes converges to the rigid-lid modes frequency  $\omega = |k|N_1/\pi n$ , and the speed of the modes becomes the speed of the baroclinic rigid-lid modes  $v = \text{sign}(k)N_1/\pi n$ .

#### c. Disappearing boundary as $N_1/N_2 \rightarrow 1$

In the singular limit  $N_1/N_2 \rightarrow 1$ , which corresponds to  $\alpha \rightarrow +\infty$ , the interface disappears and the leaky modes cease to exist. For constant  $N = N_1 = N_2$  the equation  $\phi_{zzz} = -N^2 \phi_{xx}$  factors, yielding two independent wave solutions with upward and downward group velocities, of which the effective boundary condition selects only the upward-moving one:

$$\begin{aligned} (\partial_{tz} - |k|N)(\partial_{tz} + |k|N)\phi &= 0, \\ (\partial_{tz} + |k|N)\phi &= 0 \quad \text{at } z = 1, \\ \phi &= 0 \quad \text{at } z = 0. \end{aligned}$$

#### d. The new $n = 0$ mode

We have discovered a new mode with  $n = 0$ , which is not present in the classical model. It has zero speed, decays on a time scale of 1 h times the horizontal length scale expressed in thousands of kilometers, and does not have oscillations in the vertical. Its vertical structure for any value of  $\alpha$  is simply

$$f(z) = \sinh(\alpha z).$$

In the rigid-lid limit  $\alpha \rightarrow 0$ , this mode disappears, as expected.

This is a tropospheric mode with a barotropic vertical structure, yet zero speed, which appears both in 2D with (11) and 3D with (12). This mode represents the fast adjustment of the troposphere to global (nonoscillatory) perturbations in the vertical.

#### e. The leaky-lid solutions exhibit vertical tilts

In the case of only one horizontal direction, the flow has a streamfunction  $\psi = \text{Re}\{if_n(z)e^{ikx + \lambda t}/k\}$  in the  $(x, z)$  plane, which we plot in Fig. 3 for  $n = 2$ . We see the vertical pattern of alternating local maxima and minima for the rigid-lid case (Fig. 3a), which tilts as the leak increases  $\alpha > 0$ , and eventually becomes a sequence of slanted ridges and troughs with slope  $2H/Ln$  as  $\alpha \rightarrow \infty$  (Fig. 3c). The only leaky mode with no tilt is the  $n = 0$  mode.

#### f. Interpretation

The decay time scales of the leaky modes computed above are comparable to those associated with a broad range of atmospheric phenomena. This suggests that the leakage of wave energy from the troposphere to the stratosphere could play a significant role in these phenomena, both in the determination of their amplitude by balancing the energy input that drives them, and in setting their decay time once the forcing is gone.

This article is probably not the right venue for a thorough discussion of the effect of wave leakage on individual weather configurations, which at this point would be highly speculative. Hence, we limit ourselves to enumerate a few candidate phenomena.

The  $n = 0$  mode is nonoscillatory in the vertical, suggestive of the patterns associated with deep convection

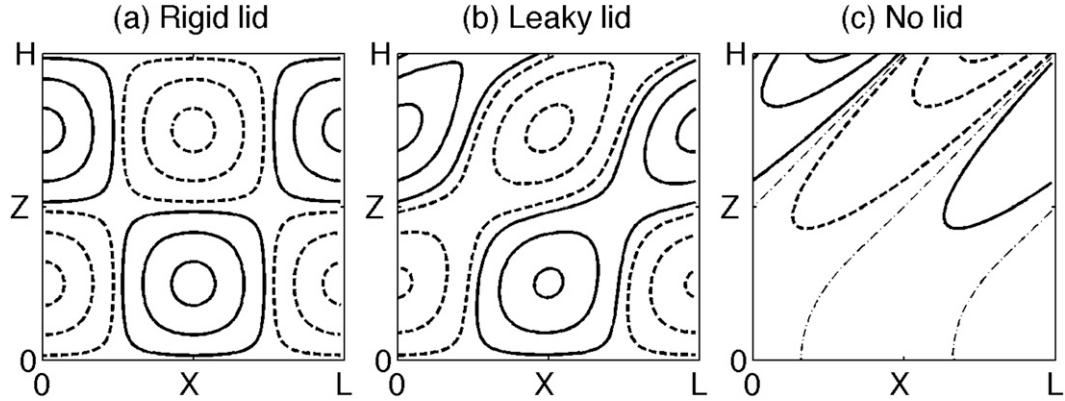


FIG. 3. Streamfunction for the  $n = 2$  mode with  $k = 2\pi/L$  for (a) the rigid lid,  $\alpha = 0$ ; (b) the leaky lid,  $\alpha = 0.5$ ; and (c) an approximation for infinite atmosphere ( $N_1/N_2 = 0.9993$ ,  $\alpha = 4$ ) with radiation boundary condition. The solid contours correspond to positive values, the dashed contours correspond to negative values, and the dash-dotted contours in (c) correspond to zero.

events. A possible scenario is the decay of mesoscale convective systems into convective vortices, for which both the  $n = 0$  and the  $n = 1$  components are significant [see Mapes (1998) for the appearance of discrete spectral bands in convective systems and Pandya and Durran (1996) and Nicholls et al. (1991) for a discussion of the interaction between gravity waves and convection in squall lines]. For typical horizontal scales of 500 km, the model predicts realistic decay time scales of 26 min for the  $n = 0$  component and 17 h for the first baroclinic ( $n = 1$ ) component of the system. Another scenario is that of stratiform precipitation, more highly baroclinic and with correspondingly longer time scales. Here, a horizontal scale of 200 km and a second baroclinic ( $n = 2$ ) mode yield a decay time about 1 day.

The tilted vertical profiles of the leaky modes also suggest that wave radiation to the stratosphere has a significant effect on equatorial wave dynamics. One candidate phenomenon is that of the convectively coupled equatorial waves. Their observed vertical structures exhibit both the vertical tilts and the localized maxima and minima, where the former are important from the perspective of the upscale momentum fluxes [Kiladis et al. (2009) and references therein]. While some models, like Mapes (2000) and Khouider and Majda (2006), suggest that the tilts result from the lower-tropospheric heating leading the middle- and upper-tropospheric heating, others, like Raymond and Fuchs (2007), argue that they are due to a moving heating source driving a wave response in addition to the upper boundary condition. We isolate the effect of wave radiation in a simple model with no moisture or heating. If the upper boundary condition is that of the leaky lid, the vertical structure simultaneously has the tilts and local maxima and minima, while the two standard choices of the rigid lid or the radiation boundary

condition capture only one feature each (Figs. 3a,c). However, the tilts are stratospheric.

## 6. Projecting onto the leaky rigid-lid modes

To complete the presentation of the model, here we show how to write the solution of the initial-value problem as a linear combination of leaky modes. This also demonstrates that the leaky modes form a complete set, and that there are no “hidden” solutions that they do not capture.

This is a nontrivial task, since the eigenvalue problem at hand is nonstandard, with the eigenvalue showing up in the boundary condition, as shown below. Furthermore, the leaky modes are not orthogonal under the standard inner product.

The approach that we have found the simplest is to map the original variable  $\phi$  and its second derivative with respect to  $t$  and  $z$  onto new variables  $A$  and  $B$ , for which the projection reduces to a Fourier series decomposition, and then map them back into  $\phi$  and  $\phi_{tz}$ .

First, we separate the horizontal dependence of the system in (13), and write  $\phi(x, z, t) = e^{ikx}\Phi(z, t)$ . Then  $\Phi$  satisfies

$$\begin{aligned}\Phi_{ttzz} &= N_1^2 k^2 \Phi, \quad 0 < z < 1, \\ \Phi_{tz} &= -\mu N_1 |k| \Phi \quad \text{at } z = 1, \\ \Phi &= 0 \quad \text{at } z = 0,\end{aligned}$$

where  $1/\mu = N_1/N_2 = \tanh(\alpha)$ . Absorbing  $N_1 |k|$  into a new time  $\tau$ , these equations simplify to

$$\begin{aligned}\Phi_{\tau\tau z z} &= \Phi, \quad 0 < z < 1, \\ \Phi_{\tau z} &= -\mu \Phi \quad \text{at } z = 1, \\ \Phi &= 0 \quad \text{at } z = 0.\end{aligned}\tag{22}$$



Notice that if we separate the time dependence in these equations through a factor  $e^{\lambda\tau}$ , then the resulting problem in  $z$  has  $\lambda$  in the boundary condition.

Motivated by the functional form of the leaky-lid modes in (17), we introduce new variables  $A$  and  $B$ , related to  $\Phi$  by the following transformation

$$\begin{pmatrix} \Phi \\ \Phi_{z\tau} \end{pmatrix} = \begin{pmatrix} -\sinh(\alpha z) & \cosh(\alpha z) \\ \cosh(\alpha z) & -\sinh(\alpha z) \end{pmatrix} \begin{pmatrix} A \\ iB \end{pmatrix}. \quad (23)$$

Under this transformation the system in (22) is equivalent to

$$\begin{aligned} A_{z\tau} &= iB + i\alpha B_{\tau}, & B_{z\tau} &= -iA - i\alpha A_{\tau}, \\ B &= 0 & \text{at } z &= 0, 1. \end{aligned} \quad (24)$$

This last system is easily solvable using cosine Fourier series for  $A$  and sine Fourier series for  $B$ , with the standard formulas to compute the coefficients. Furthermore, through the transformation from  $(A, B)$  to  $\Phi$ , the above Fourier series become the leaky modes for  $\Phi$ .

## 7. Conclusions

We offer a potential answer to the debate on whether the atmosphere should be modeled as infinite or finite, and whether there exist discrete modes at all: the troposphere can be studied in isolation, but with an effective boundary condition at the top that allows a fraction of the energy in the long waves to escape into the stratosphere. This approach is valid under the assumption that the waves that escape through the tropopause do not return after being reflected at stratospheric inhomogeneities. Our effective boundary condition gives the same reflection coefficient at the tropopause as if there were a stratosphere above, with a prescribed buoyancy frequency. This self-contained model of the troposphere has the dissipative properties associated with the upward wave radiation of an infinite atmosphere, even though it has a discrete spectrum in the vertical. The new leaky rigid-lid modes, which we compute assuming that the buoyancy frequency is piecewise constant, have tilted vertical profiles, and decay with time scales comparable to the observed relaxation times of some atmospheric phenomena. This suggests that upward wave radiation could be a key player in these phenomena and provides a modeling framework to study them.

In this article, we have concentrated on the new physics and mathematical formulation of the leaky rigid lid, for which we have adopted the simplest scenario of linear and irrotational waves. Further work is required

to include the effects of both vorticity and Earth rotation. This would constitute an important step in the study of the interplay between wave radiation and eddies and storms.

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